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Selecting a Minimax Estimator Doing Well at a Point*

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1. INTRODUCTION

Let $X_i \sim N(\theta_i, I_p)$, $i = 1, 2,..., n$ $(n > p+1, p > 1)$ be an independent finite sequence of p-dimensional vectors. Let $X = (X_1,..., X_n)$ be $p \times n$ matrix of random variables with the matrix of means $\theta = (\theta_1, \theta_2,..., \theta_n)$. Efron and Morris [2] generalized the James–Stein estimator: they proved that $X - (n-p-1)\, S^{-1}X$ (along with its positive part $(I - (n-p-1)\, S^{-1})_+\, X$) is a minimax estimator. Stein [3] considered the estimators of form $X + \nabla \ln f(l)$, where $l = (l_1, l_2,..., l_p)^{\mathrm{T}}$ is a vector with components $l_1 \geqslant l_2 \geqslant \cdots l_p \geqslant 0$, the characteristic roots of $S = XX^T$, and $\nabla \ln f(l) = ((\partial/\partial X_{jk}) \ln f(l))$ is the matrix of partial derivatives of the function $\ln f(l)$. He gave a condition under which the corresponding estimator $X + \nabla \ln f(l)$ is minimax. Zheng [4] gave a class of minimax estimators of the form $X + \nabla \ln f(l)$, where

$$f(l) = \prod_{i=1}^{p} \int_{l_i}^{+\infty} \frac{\mu_i(t_i)\, dt_i}{t_i^{1/2}(n-p+1)}. \tag{1.1}$$

But when $p > 1$, $n > p+1$, the estimator $\delta(X) = X + \nabla \ln f(l)$, where $f(l)$ is given by (1.1), is inadmissible. For fixed $f(l)$ of form (1.1), Zheng [4] found a class of estimators of the form

$$\delta_\beta(X) = X + \nabla \ln f(l) - 2X \Big/ \left(\frac{\operatorname{tr} S}{p(p+1) - 2} + \beta(l) \right) \tag{1.2}$$

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where $\beta(l)$ is a function belonging to the set

$B = \beta(l)$:

$$\begin{aligned}
&\text{(i)} \quad \sum_{i=1}^{p} (\partial/\partial l_i)\, \beta(l) \cdot l_i \leqslant 0; \\
&\text{(ii)} \quad E_\theta\{(\partial/\partial l_i)\, \beta(l) \cdot \sqrt{l_i}\} < +\infty,\ i = 1, 2, ..., p; \qquad (1.3) \\
&\text{(iii)} \quad \beta(l) \geqslant 0, \text{ and } \beta(l) > 0 \text{ on a set of positive measure.}
\end{aligned}$$

For every $\beta \in B$, $\delta_\beta(X)$ improves upon the minimax estimator $\delta(X) = X + \nabla \ln f(l)$. Actually, we can slightly enlarge the class B without changing the original proof given in [4]. Let

$B_1 = \beta(l)$:

$$\begin{aligned}
&\text{(i)} \quad E_\theta\{(\partial/\partial l_i)\, \beta(l) \cdot \sqrt{l_i}\} < \infty,\ i = 1, 2, ..., p; \\
&\text{(ii)} \quad (p(p+1)/2)\, \beta(l) - \sum_{i=1}^{p} l_i(\partial/\partial l_i)\, \beta(l) \geqslant 0; \\
&\text{(iii)} \quad L\left\{X : \tfrac{1}{2}p(p+1)\, \beta(l) - \sum_{i=1}^{p} l_i(\partial/\partial l_i)\, \beta(l) > 0\right\} > 0; \qquad (1.4) \\
&\text{(iv)} \quad \beta(l) \geqslant 0.
\end{aligned}$$

Then for $\beta(l) \in B_1$, the estimator $\delta_\beta(X)$ given by (1.2) improves upon the estimator $\delta(X) = X + \nabla \ln f(l)$.

The purpose of this paper is to select an estimator $\delta^*(X)$ which dominates $\delta(X) = X + \nabla \ln f(l)$ such that the estimator is the best at $\theta = 0$ among estimators $\delta_\beta(X) = \delta(X) - 2X/(\operatorname{tr} S/(p(p+1) - 2) + \beta(l))$, $\beta \in B_1$; i.e., δ^* satisfies

$$\text{(i)} \quad E_\theta\{\| \delta^* - \theta \|^2\} \leqslant E_\theta\{\| X + \nabla \ln f(l) - \theta \|^2\} \tag{1.5}$$

$$\text{(ii)} \quad E_{\theta=0}\{\| \delta^* - \theta \|^2\} = \inf_{\beta \in \beta_1} E_{\theta=0}\, \{\| \delta_\beta - \theta \|^2\} \tag{1.6}$$

where $\| \delta^* - \theta \|^2 = \operatorname{tr}(\delta^* - \theta)^{\mathrm{T}}\, (\delta^* - \theta)$, etc. Notice that δ^* does not have the form of δ_β for some $\beta \in B_1$. We will prove that δ^* has the form

$$\delta^*(X) = X + \nabla \ln f(l) - 2X \cdot g(X) \tag{1.7}$$

where

$$g(X) = \frac{1}{\operatorname{tr} S} \cdot \min\left\{\frac{1}{2}(\operatorname{tr} S + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}}\, X)_+,\ p(p+1) - 2\right\}. \tag{1.8}$$

In (1.1), let $\mu_1(t)=\mu_2(t)=\cdots=\mu_p(t)=1$. Then $\delta(X)=X+\nabla \ln f(l)=X-(n-p-1)\,S^{-1}X$, and

$$\delta^*(X)=X-(n-p-1)\,S^{-1}\,X-2Xg_1(X) \tag{1.9}$$

where

$$g_1(X)=\frac{1}{\operatorname{tr} S}\min\left\{p(p+1)-2, \frac{1}{2}(\operatorname{tr} S-p(n-p-1))_+\right\}. \tag{1.10}$$

In (1.1), let $\mu_1(t)=\mu_2(t)=\cdots=\mu_p(t)=\mu(t)$, where

$$\begin{aligned}\mu(t)&=1 \qquad \text{if} \quad t>n-p-1\\ &=(n-p-1)^{-(1/2)(n-p+1)}\,t^{(1/2)(n-p+1)}e^{(1/2)(n-p-1-t)}\\ &\qquad\quad \text{if} \quad t\leqslant n-p-1.\end{aligned} \tag{1.11}$$

Then, $\delta(X)=(I-(n-p-1)\,S^{-1})_+\,X$, and

$$\delta^*(X)=(I-(n-p-1)\,S^{-1})_+\,X-2X\cdot g_2(X) \tag{1.12}$$

where

$$g_2(X)=\frac{1}{\operatorname{tr} S}\min\left\{p(p+1)-2, \frac{1}{2}\sum_{i=1}^{p}(l_i-(n-p-1))_+\right\}. \tag{1.13}$$

2. The Main Result

Suppose $\mu_1(t),..., \mu_p(t)$ are nondecreasing functions satisfying the condition of Theorem 1 of [4], i.e.,

(i) $\mu_{i+1}(t)/\mu_i(t)$ nondecreasing, $i=0, 1, 2,..., p-1$,

$$\left(\mu\equiv 1, \frac{0}{0}\overset{\text{def}}{=}1, \frac{1}{0}\overset{\text{def}}{=}+\infty\right). \tag{2.1}$$

(ii) $\displaystyle\int_a^{+\infty}\mu_i(t)\,dt/t^{(1/2)(n-p+1)}<+\infty \ (\forall a>0),\ i=1, 2,..., p.$ (2.2)

Then according to Theorem 1 of [4], $X+\nabla \ln f(l)$ is a minimax estimator of θ, where $f(l)$ is given by (1.1). Now we have

THEOREM 1. *Suppose that $\mu_1(t), \mu_2(t), \ldots, \mu_p(t)$ are absolutely continuous, nondecreasing functions satisfying* (2.1), (2.2) *and that the functions of* v,

$$v - \frac{2\mu_i(v)\, v}{v^{(1/2)(n-p+1)}} \Big/ \int_v^{+\infty} \frac{\mu_i(t)\, dt}{t^{(1/2)(n-p+1)}} \qquad i = 1, 2, \ldots, p, \tag{2.3}$$

are nondecreasing functions of v. Then the estimator $\delta^*(X)$ given by (1.7) satisfies (1.5) and (1.6). In addition, we have the following:

$$\begin{aligned} E_{\theta=0}\{\|\delta^*(X) - \theta\|^2\} &= E_{\theta=0}\{\|X + \nabla \ln f(l) - \theta\|^2\} \\ &\quad - 4E_{\theta=0}\left\{\frac{(\operatorname{tr} S + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X) \operatorname{tr} S^{-(1/2)((p+1)}}{\operatorname{tr} S^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_0}\right. \\ &\quad \left. - \frac{\operatorname{tr} S^{-p(p+1)+1}}{((\operatorname{tr} S)^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_0)^2}\right\} \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \psi_0 &= +\infty, && \text{if } \operatorname{tr} S + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X \leqslant 0; \\ &= \left(\frac{2 \operatorname{tr} S^{-(1/2)p(p+1)+1}}{\operatorname{tr} S + \operatorname{tr} \nabla \ln f(l)^T X} - \frac{\operatorname{tr} S^{-(1/2)p(p+1)+1}}{p(p+1)-2}\right)_+ \\ & && \text{if } \operatorname{tr} S + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X > 0. \end{aligned} \tag{2.5}$$

Proof. Let

$$\Delta(\delta^*, \theta) = E_\theta\{\|X + \nabla \ln f(l) - \theta\|^2 - \|\delta^* - \theta\|^2\}. \tag{2.6}$$

First, we prove that (1.5) holds. Using (1.7), we get

$$\begin{aligned} \Delta(\delta^*, \theta) &= 4E_\theta\{g(X) \operatorname{tr} X^{\mathrm{T}}(X - \theta)\} - 4E_\theta\{g^2(X)\|X\|^2\} \\ &\quad + 4E_\theta\{g(X) \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l)\}. \end{aligned} \tag{2.7}$$

Using integration by parts,

$$E_\theta\{g(X) \operatorname{tr} X^{\mathrm{T}}(X - \theta)\} = npE_\theta\{g(X)\} + \sum_{j=1}^{p} \sum_{k=1}^{n} E_\theta\left\{\frac{\partial g}{\partial X_{jk}} \cdot X_{jk}\right\}. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\begin{aligned} \Delta(\delta^*, \theta) = 4E_\theta\Bigg\{& npg(X) + \sum_{j=1}^{p} \sum_{k=1}^{n} X_{jk} \frac{\partial g(X)}{\partial X_{jk}} \\ & - g^2(X) \operatorname{tr} S + g(X) \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l)\Bigg\}. \end{aligned} \tag{2.9}$$

First, suppose that $0<\frac{1}{2}(\operatorname{tr} S+\operatorname{tr}(\nabla \ln f(l)^{\mathrm{T}} X))<p(p+1)-2$. Then, according to (1.8), $g(X)=(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X)/(2 \operatorname{tr} S)$. Hence

$$
\begin{aligned}
&\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{\partial g}{\partial X_{jk}} \cdot X_{jk} \\
&\quad=\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{1}{2 \operatorname{tr} S} \frac{\partial}{\partial X_{jk}}(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X) \cdot X_{jk} \\
&\qquad-\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X}{2(\operatorname{tr} S)^{2}} \cdot 2 X_{jk}^{2} \\
&\quad=\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{1}{2 \operatorname{tr} S} \frac{\partial}{\partial X_{jk}}(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X) X_{jk} \\
&\qquad-\frac{1}{\operatorname{tr} S}(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X).
\end{aligned} \tag{2.10}
$$

From (4.24) of [3],

$$
\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X=-2 \sum_{i=1}^{p} \frac{\mu_{i}(l_{i})}{l_{i}^{(1/2)(n-p-1)}} \Big/ \int_{l_{i}}^{+\infty} \frac{\mu_{i}(t)\, dt}{t^{(1/2)(n-p+1)}}, \tag{2.11}
$$

so (using (4.24) of [3] again)

$$
\begin{aligned}
&\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{1}{2 \operatorname{tr} S} \frac{\partial}{\partial X_{jk}}(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X) X_{jk} \\
&\quad=\frac{1}{2 \operatorname{tr} S} \sum_{i=1}^{p}\left(l_{i}-2 \frac{\mu_{i}(l_{i})}{l_{i}^{(1/2)(n-p-1)}} \Big/ \int_{l_{i}}^{+\infty} \frac{\mu_{i}(t)\, dt}{t^{(1/2)(n-p+1)}}\right)_{l_{i}}^{\prime} \cdot 2 l_{i}.
\end{aligned}
$$

Substituting the above into (2.10), we get

$$
\begin{aligned}
\sum_{j=1}^{p} \sum_{k=1}^{n} \frac{\partial g}{\partial X_{jk}} \cdot X_{jk}=&\frac{1}{\operatorname{tr} S} \sum_{i=1}^{p}\left(l_{i}-\frac{2 \mu_{i}(l_{i})}{l_{i}^{(1/2)(n-p-1)}} \Big/ \int_{l_{i}}^{+\infty} \frac{\mu_{i}(t)\, dt}{t^{(1/2)(n-p+1)}}\right)_{l_{1}}^{\prime} \cdot l_{i} \\
&-\frac{1}{\operatorname{tr} S}(\operatorname{tr} S+\operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X).
\end{aligned}
$$

So when $0<\frac{1}{2}(\operatorname{tr} S+\operatorname{tr}(\nabla \ln f(l)^{\mathrm{T}} X))<p(p+1)-2$, the following holds:

$$
\begin{aligned}
&n p g(X)+\sum_{j=1}^{p} \sum_{k=1}^{n} X_{jk} \frac{\partial g}{\partial X_{jk}}-g^{2}(X) \cdot \operatorname{tr} S+g(X) \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l) \\
&\quad=g(X)(p(n-p-1)+\operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l))
\end{aligned}
$$

$$+\frac{1}{\operatorname{tr} S}\sum_{i=1}^{p}\left(l_i-\frac{2\mu_i(l_i)}{l_i^{(1/2)(n-p-1)}}\Big/\int_{l_i}^{+\infty}\frac{\mu_i(t)\,dt}{t^{(1/2)(n-p+1)}}\right)'_{l_i}\cdot l_i$$
$$+\frac{\operatorname{tr} S+\operatorname{tr}\nabla\ln f(l)^{\mathrm{T}}X}{\operatorname{tr} S}\left[\frac{p(p+1)}{2}-1-\frac{1}{4}(\operatorname{tr} S+\operatorname{tr}\nabla\ln f(l)^{\mathrm{T}}X)\right]$$
$$\geqslant 0. \tag{2.12}$$

Second, suppose that $\frac{1}{2}(\operatorname{tr} S+\operatorname{tr}(\nabla\ln f(l)^{\mathrm{T}}X))>p(p+1)-2$. In this case, $g(X)=(1/\operatorname{tr} S)(p(p+1)-2)$. Then,

$$\sum_{j,k}\frac{\partial g}{\partial X_{jk}}\cdot X_{jk}=-2g(X).$$

So,

$$npg(X)+\sum_{j=1}^{p}\sum_{k=1}^{n}\frac{\partial g}{\partial X_{jk}}X_{jk}-g^2(X)\operatorname{tr} S+g(X)\operatorname{tr} X^{\mathrm{T}}\nabla\ln f(l)$$
$$=(p(n-p-1)+\operatorname{tr} X^{\mathrm{T}}\nabla\ln f(l))\,g(X)\geqslant 0. \tag{2.13}$$

Finally, suppose that $\frac{1}{2}(\operatorname{tr} S+\operatorname{tr}\nabla\ln f(l)^{\mathrm{T}}X)\leqslant 0$. In this case $g(X)\equiv 0$. Then,

$$npg(X)+\sum_{j,k}\frac{\partial g}{\partial X_{jk}}X_{jk}-g^2(X)\operatorname{tr} S+g(X)\operatorname{tr} X^{\mathrm{T}}\nabla\ln f(l)\equiv 0. \tag{2.14}$$

From (2.12), (2.13), (2.14), and (2.9), we get (1.5).

To prove (1.6, we first prove the inequality

$$\Delta(\delta_\beta,0)\leqslant 4E_{\theta=0}\left\{\frac{(\operatorname{tr} S+\operatorname{tr}\nabla\ln f(l)^{\mathrm{T}}X)(\operatorname{tr} S)^{-(1/2)p(p+1)}}{((\operatorname{tr} S)^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi_0)}\right.$$
$$\left.-\frac{(\operatorname{tr} S)^{-p(p+1)+1}}{((\operatorname{tr} S)^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi_0)^2}\right\}\quad\forall\beta\in B_1 \tag{2.15}$$

where ψ_0 is given by (2.5) and δ_β is given by (1.2).

According to Theorem 4 of [4], $\delta_\beta(X)(=X+\nabla\ln f(l)-2X/(\operatorname{tr} S/(p(p+1)-2)+\beta(l)))$ dominates the estimator $X+\nabla\ln f(l)$ and

$$\Delta(\delta_\beta,\theta)=8E_\theta\left\{\frac{(1/2)p(p+1)\,\beta(l)-\sum_{i=1}^{p}l_i(\partial/\partial l_i)\,\beta(l)}{(\operatorname{tr} S/(p(p+1)-2)+\beta(l))^2}\right\}$$
$$+4E_\theta\left\{\frac{p(n-p-1)+\operatorname{tr} X^{\mathrm{T}}\nabla\ln f(l)}{\operatorname{tr} S/(p(p+1)-2)+\beta(l)}\right\} \tag{2.16}$$

where $\beta \in B_1$. Considering the point $\theta = 0$, since the distribution of $l = (l_1, ..., l_p)$ is known [1], we get

$$\Delta(\delta_\beta, 0) = \iint_{l_1 \geqslant \cdots \geqslant l_p \geqslant 0} \cdots \int \left[\frac{8((1/2)p(p+1)\,\beta(l) - \sum_{i=1}^p l_i(\partial\beta(l)/\partial l_i))}{(\operatorname{tr} S/(p(p+1)-2) + \beta(l))^2} + \frac{4(p(n-p-1) + \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l))}{\operatorname{tr} S/(p(p+1)-2) + \beta(l)} \right] \times \frac{\pi^{(1/2)p} \prod_{i=1}^p l_i^{(1/2)(n-p-1)} e^{-(1/2)\sum_{i=1}^p l_i} \prod_{i<j} (l_i - l_j)}{2^{(1/2)pn} \prod_{i=1}^p \{\Gamma(\frac{1}{2}(n+1-i))\, \Gamma(\frac{1}{2}(p+1-i))\}}\, dl_1 \cdots dl_p.$$

Let

$$\begin{aligned} u &= l_1 + \cdots + l_p, \\ \Delta_1 &= (l_1 - l_2)/u, \\ \Delta_2 &= (l_2 - l_3)/u, \\ &\vdots \\ \Delta_{p-2} &= (l_{p-2} - l_{p-1})/u, \\ \Delta_{p-1} &= l_{p-1}/u. \end{aligned} \tag{2.17}$$

Then,

$$\Delta(\delta_\beta, 0) = \int_{\mathscr{D}} \cdots \iiint_0^\infty \left[\frac{8((1/2)\,p(p+1)\,\beta(l) - \sum_{i=1}^p l_i(\partial\beta(l)/\partial l_i))}{(u/(p(p+1)-2) + \beta(l))^2} + \frac{4(p(n-p-1) + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X)}{u/(p(p+1)-2) + \beta(l)} \right] \times u^{(np/2)-1} e^{-u/2} du \cdot D(\Delta_1, ..., \Delta_{p-1})\, d\Delta_1, ..., d\Delta_{p-1}$$

where

$$\mathscr{D} = \{(\Delta_1, ..., \Delta_{p-1}) \colon \Delta_i \geqslant 0,\ i = 1, ..., p-1,\ 0 < 1 - (\Delta_1 + 2\Delta_2 + \cdots + (p-1)\,\Delta_{p-1}) < \Delta_{p-1}\}$$

and $D(\Delta_1, ..., \Delta_{p-1})$ is a function of $\Delta_1, ..., \Delta_{p-1}$. Denote $\beta(l) = r(u, \Delta_1, ..., \Delta_{p-1})$, and it is easy to show that $\sum_{i=1}^p l_i(\partial\beta(l)/\partial l_i) = r'_u \cdot u$. So,

$$\Delta(\delta_\beta, 0) = \int_{\mathscr{D}} \cdots \int D(\Delta_1, ..., \Delta_{p-1})\, d\Delta_1, ..., d\Delta_{p-1} \times \int_0^\infty \left(\frac{8((1/2)\,p(p+1)\,r - r'_u \cdot u)}{(u/(p(p+1)-2) + r)^2} + 4 \cdot \frac{p(n-p-1) + \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l)}{u/(p(p+1)-2) + r} \right) \cdot u^{np/2-1} e^{-u/2}\, du.$$

Consider the integration

$$I=\int_0^\infty\left(\frac{8((p(p+1)/2)\,r-r'_u\cdot u)}{(u/(p(p+1)-2)+r)^2}+4\cdot\frac{p(n-p-1)+\operatorname{tr}X^{\mathrm{T}}\nabla\ln f(l)}{(u/(p(p+1)-2)+r)}\right)u^{np/2-1}e^{-u/2}\,du.$$

Let $\psi(u,\Delta_1,\dots,\Delta_{p-1})=r\cdot u^{-(1/2)p(p+1)}$. Then,

$$I=I(\psi)=-8\int_0^\infty\frac{\psi'_u u^{(1/2)p(n-p-1)}e^{-u/2}\,du}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi)^2}+4\int_0^\infty\frac{p(n-p-1)+\operatorname{tr}X^{\mathrm{T}}\nabla\ln f(l)}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi)}\times e^{-u/2}u^{(p/2)(n-p-1)-1}\,du.$$

Using integration by parts, we get

$$I(\psi)=-8\int_0^\infty\frac{u^{(1/2)pn-p(p+1)}\,e^{-u/2}\,du}{2(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi)^2}+4\int_0^\infty\frac{u+\operatorname{tr}X^{\mathrm{T}}\nabla\ln f(l)}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi)}\,e^{-u/2}u^{(1/2)p(n-p-1)-1}\,du.$$

Considering the problem of maximizing I, we get

$$I(\psi)\leqslant I(\psi_0)\tag{2.18}$$

where ψ_0 is given by (2.5). Hence

$$\Delta(\delta_\beta,0)\leqslant\int\cdots\int_{\mathscr{D}}D(\Delta_1,\dots,\Delta_{p-1})\,d\Delta_1,\dots,d\Delta_{p-1}\cdot\times\int_0^\infty\left[\frac{4(u+\operatorname{tr}\nabla\ln f(l)^{\mathrm{T}}X)\,u^{-p(p+1)/2}}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi_0)}-\frac{4u^{-p(p+1)+1}}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2)+\psi_0)^2}\right]u^{+pn/2-1}e^{-u/2}\,du.$$

From the above we know that (2.15) holds.

Let

$$\psi_m=\psi_0\cdot I_{\{\psi_0\leqslant m\}}+m\cdot I_{\{\psi_0>m\}}\tag{2.19}$$

where ψ_0 is given by (2.5). Let

$$g_m(X) = 1\bigg/\left(\frac{\operatorname{tr} S}{p(p+1)-2} + (\operatorname{tr} S)^{(1/2)p(p+1)} \cdot \psi_m\right)$$
$$= 1\bigg/\left(\frac{\operatorname{tr} S}{p(p+1)-2} + \beta_m(l)\right) \tag{2.20}$$

and

$$\delta_{\beta_m}(X) = X + \nabla \ln f(l) - 2Xg_m(X). \tag{2.21}$$

Since $\beta_m(l) = (\operatorname{tr} S)^{p(p+1)/2} \cdot \psi_m$, it is easy to verify that

$$\frac{p(p+1)}{2}\beta_m(l) - \sum_{i=1}^{p} \frac{\partial \beta_m(l)}{\partial l_i} = -\frac{\partial \psi_m}{\partial u} u^{(1/2)p(p+1)+1} \tag{2.22}$$

where $(u, \Delta_1, \ldots, \Delta_{p-1})$ and $(l_1, \ldots, l_p)$ are connected by (2.17).

Note that $\beta_m \in B_1$. The technical verification is omitted. According to (2.16) and (2.22),

$$\Delta(\delta_{\beta_m}, 0) = 8E_{\theta=0}\left\{\frac{-(\partial\psi_m/\partial u)\, u^{(1/2)p(p+1)+1}}{(u/(p(p+1)-2) + \psi_m u^{(p(p+1))/2})^2}\right\}$$
$$+ 4E_{\theta=0}\left\{\frac{p(n-p-1) + \operatorname{tr} \nabla \ln f(l)^{\mathrm{T}} X}{(u/(p(p+1)-2) + \psi_m u^{(p(p+1))/2})}\right\}.$$

After using integration by parts, we get

$$\Delta(\delta_{\beta_m}, 0) = 4\int\cdots\int_{\mathscr{D}} D(\Delta_1, \ldots, \Delta_{p-1})\, d\Delta_1, \ldots, d\Delta_{p-1}$$
$$\times \int_0^{\infty}\left(\frac{(u + \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l))\, u^{-(1/2)p(p+1)}}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_m)}\right.$$
$$\left. - \frac{u^{-p(p+1)+1}}{(u^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_m)^2}\right)$$
$$\times u^{pn/2-1} e^{-u/2}\, du. \tag{2.23}$$

Using the Dominated Convergence Theorem, we get

$$\lim_{m \to +\infty} \Delta(\delta_{\beta_m}, 0) = 4E_{\theta=0}\left\{\frac{(\operatorname{tr} S + \operatorname{tr} X^{\mathrm{T}} \nabla \ln f(l)) \operatorname{tr} S^{-(1/2)p(p+1)}}{(\operatorname{tr} S)^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_0}\right.$$
$$\left. - \frac{(\operatorname{tr} S)^{-p(p+1)-1}}{((\operatorname{tr} S)^{-(1/2)p(p+1)+1}/(p(p+1)-2) + \psi_0)^2}\right\}. \tag{2.24}$$

On the other hand, from the definition of δ^* and δ_{β_m}, it is easy to show that $\lim_{m\to+\infty}\delta_{\beta_m}=\delta^*$. Hence, after using the Dominated Convergence Theorem, we get

$$\lim_{m\to+\infty}\Delta(\delta_{\beta_m},0)=\Delta(\delta^*,0). \tag{2.25}$$

Comparing (2.25) with (2.24), we get (2.4). Since we have proved (2.15) it means that (1.6) holds. ▮

Using the result of Theorem 1, when $\mu_1(t)=\cdots=\mu_p(t)\equiv 1$, the estimator $\delta^*(X)$ given by (1.9) dominates the crude Efron–Morris estimator $X-(n-p-1)\,S^{-1}X$ and is the best at $\theta=0$ among the class $\{\delta_\beta=X+(n-p-1)\,S^{-1}X-2X/(\operatorname{tr}S/(p(p+1)-2)+\beta(l)): \beta\in B_1\}$, when $\mu_1(t)=\cdots=\mu_p(t)\equiv\mu$ given by (1.11), the corresponding estimator $\delta^*(X)$ given by (1.12) dominates the Efron–Morris estimator $(I-(n-p-1)\,S^{-1})_+\,X$ and is the best at $\theta=0$ among the class $\{\delta_\beta=(I-(n-p-1)\,S^{-1})_+\,X-2X/(\operatorname{tr}S/(p(p+1)-2)+\beta): \beta\in B_1\}$.

References

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis.* Wiley, New York.

[2] Efron, B. and Morris, C. (1972). Empirical Bayes on vector observations: An extension of Stein's method. *Biometrika* **59** (2) 335.

[3] Stein, C. (1974). Estimation of the mean of a multivariate normal distribution. In *Proc. Prague Sympos. Asymptotic Statist.*, pp. 345–381.

[4] Zheng, Z. (1986). On estimation of matrix of normal mean. *J. Multivar. Anal.* **18**, 70–82.